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# Enhancement of Signal-to-Noise Ratio

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It is shown that the ratio of signal-to-noise ratio at the output to that at the input can be made larger than unity when a nonlinear system is subjected to a weak periodic perturbation and a harmonic noise. This theoretical result is in conformity with the experimental observation published recently by R. Li *et al.*

**KEY WORDS:** Harmonic noise; stochastic resonance; projected process; signal-to-noise ratio.

## 1. INTRODUCTION

There has been an upsurge of interest (both theoretical and experimental) in the topic of enhancement of signal-to-noise ratio, more commonly known as stochastic resonance, in recent years.[1-15] It is a noise-induced large response to a weak periodic signal. The system in a noisy environment is influenced by a periodic force (signal) and also by a conservative force which should be nonlinear in nature in order to get the enhancement of the signal at the output. At the output, the signal is recognized by its original frequency. The enhancement of signal power implies that some portion of the incoherent noise is fed into the coherent signal power. Thus a cooperative effect of noise and periodic forcing in a nonlinear system takes place, which makes this problem highly interesting to the nonlinear science community also.

Instead of observing the signal at the output, the more relevant quantity is the amplification of signal over noise or signal-to-noise ratio (SNR) at the output. As the SNR represents the quality of the signal, which plays a central role in the information transfer, current effort is directed toward searching for a device where a "better" SNR could be achieved. The qualification of "better" is, however, fixed[15] by looking at the ratio of

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SNR at the output to the SNR at the input. Thus, mere enhancement of signal-to-noise power at the output is not of concern, but whether the SNR at the output can be made larger than that of the input.

Most of the systems analyzed so far consist of a bistable system influenced by a weak priodic signal and white noise. It has been noted[15] that in these systems the ratio we are looking at is less than unity. Recently, an experiment[15] has been carried out where instead of white noise, harmonic noise was used at the input. Noise generated by a damped system driven harmonically and by white noise as its source is called harmonic noise. Their system,[15] however, is not coupled directly by a weak signal. The signal instead is applied at the source of the noise generator. Experiment shows that the ratio of the output SNR to the input SNR can be made larger than unity. In this paper we analyze this system theoretically.

In Section 2 we briefly sketch the theory of enhancement of signal-to-noise ratio for a nonlinear system driven by a weak periodic signal and white noise and show that the ratio of the output SNR to the input is less than unity. In Section 3 we analyze theoretically the system used in the experiment[15] and obtain an expression for the ratio of the output SNR to the input SNR and show that this could be made larger than unity. Finally, a few concluding remarks are given in Section 4.

## 2. THEORY OF ENHANCEMENT OF SIGNAL-TO-NOISE RATIO

The stochastic system that is being considered now is nonlinear and driven by a weak periodic signal and white noise. The Langevin equation describing this system is

$$\dot{x} = -V_0'(x) + A\cos(\Omega t + \theta) + \Gamma(t) \tag{2.1}$$

where $V_0(x)$ is the potential and $-V_0'(x)$ is the conservative force derived from the potential $V_0(x)$. The force $V_0'(x)$ is assumed to be nonlinear. Some other characteristic features of $V_0(x)$ will be specified whenever they are required in the further analysis. Here, $A$ is the amplitude of the periodic signal of period $2\pi/\Omega$. The amplitude $A$ is assumed to be small and $\theta$ is an arbitrary phase. $\Gamma(t)$ is a white noise specified by

$$\langle \Gamma(t) \rangle = 0 \tag{2.2a}$$

$$\langle \Gamma(t)\, \Gamma(t') \rangle = 2D\, \delta(t - t') \tag{2.2b}$$

where angular brackets denote the average over ensembles and $D$ is the diffusion constant, a direct measure of the strength of the noise.

The corresponding Fokker–Planck equation for Eq. (2.1) is written in the convenient form

$$\frac{\partial P(x, t)}{\partial t} = (L^{(0)} + L^{(1)})\, P(x, t) \tag{2.3}$$

where $P(x, t)$ is the probability distribution function and the probability current for the unperturbed (when the signal is absent) system $S^{(0)}(x, t)$ is related to $L^{(0)}$ as

$$-\frac{\partial S^{(0)}(x, t)}{\partial x} = L^{(0)}(x)\, P(x, t) \tag{2.4}$$

and $L^{(0)}$ is given by

$$L^{(0)}(x) = D\frac{\partial}{\partial x}\, e^{-\Phi(x)}\frac{\partial}{\partial x}\, e^{\Phi(x)} \tag{2.5}$$

with

$$\Phi(x) = V_0(x)/D \tag{2.6}$$

The perturbation operator $L^{(1)}$ is given by

$$L^{(1)}(x, t) = -A\cos(\Omega t + \theta)\frac{\partial}{\partial x} \tag{2.7}$$

We wish to analyze the system in terms of the perturbation to the unperturbed states; thus a spectral decomposition of $L^{(0)}$ is required. We note, however, that the operator $L^{(0)}$ is not Hermitian, although $e^{\Phi}L^{(0)}$ or $e^{\Phi/2}L^{(0)}e^{-\Phi/2}$ is Hermitian. Hence we transform Eq. (2.3) into a more convenient form which contains the Hermitian operator as its unperturbed part. The transformed equation reads

$$\frac{\partial \bar{P}(x, t)}{\partial t} = (\bar{L}^{(0)} + \bar{L}^{(1)})\, \bar{P}(x, t) \tag{2.8}$$

where the Hermitian operator $\bar{L}^{(0)}$ is given by

$$\bar{L}^{(0)}(x) = De^{\Phi/2}\frac{\partial}{\partial x}\, e^{-\Phi(x)}\frac{\partial}{\partial x}\, e^{\Phi/2} \tag{2.9}$$

The probability current expressed in terms of the new function $\bar{P}$ is

$$\bar{S}^{(0)}(x) = -De^{-\Phi(x)} \frac{\partial}{\partial x} e^{\Phi(x)/2} \bar{P} \tag{2.10}$$

and the perturbation operator $\bar{L}^{(1)}$ is

$$\bar{L}^{(1)} = -A\cos(\Omega t + \theta)\,[\partial/\partial x - V_0'(x)/2D] \tag{2.11}$$

The transformed distribution $\bar{P}(x, t)$ is related to $P(x, t)$ by

$$\bar{P}(x, t) = e^{\Phi/2} P(x, t) \tag{2.12}$$

Let the complete set of orthonormal eigenfunctions and corresponding eigenvalues of the operator $\bar{L}^{(0)}$ be denoted as $\{|n\rangle\}$ and $\{-\lambda_n\}$, $n = 0, 1, 2, \ldots$, respectively, such that

$$\bar{L}^{(0)}\,|n\rangle = -\lambda_n\,|n\rangle \tag{2.13}$$

with

$$\langle n|m\rangle = \delta_{nm} \tag{2.14}$$

As $\langle n|\,\bar{L}^{(0)}\,|n\rangle$ for arbitrary $n$ can always be expressed as a negative-semi-definite form, one concludes that $\lambda_n \geqslant 0$, $\forall n$. We order them such that $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, and assume that the potential $V_0(x) \to \alpha$ as $|x| \to \alpha$ at least as $|x|^\alpha$ with $\alpha > 1$. This assumption implies that $\lambda_1 \ll \lambda_2, \lambda_n$, which immensely simplifies the calculation, yet brings out the enhancement, as we will see. Out of all these eigenstates, a particularly simple one is the lowest one corresponding to $\lambda_0 = 0$. This state can be easily found by looking at the operator $L^{(0)}(x)$ in Eq. (2.5). Thus the ground state $\langle x|0\rangle$ for the operator $\bar{L}^{(0)}$ with natural boundary conditions is written with the help of the definition (2.12) as

$$\bar{\psi}_0(x) = \langle x|0\rangle = N_0 e^{-\Phi(x)/2}$$

where $N_0$ is the normalization constant and $\Phi(x)$ is given by Eq. (2.6).

The conditional probability $\bar{P}(x, t\,|\,x_0, t_0)$ can then be written in terms of the conditional probability $\bar{P}_0(x, t\,|\,x_0, t_0)$ for the unperturbed problem (with $A = 0$) as

$$\bar{P}(x, t\,|\,x_0, t_0) = \bar{P}_0(x, t\,|\,x_0, t_0) + \int_{t_0}^{t} dt_1 \int dx_1\, \bar{P}_0(x, t\,|\,x_1, t_1)\, \bar{L}^{(1)}(x_1, t_1) \times \bar{P}_0(x_1, t_1\,|\,x_0, t_0) \tag{2.15}$$

where $\bar{L}^{(1)}$ is the perturbation operator containing the amplitude $A$, which is small (weak signal), such that we keep terms proportional to $A$ only and neglect terms $O(A^2)$ in the spirit of linear response theory. This is why we keep term proportional to $A$ in Eq. (2.15).

The conditional probability for the unperturbed operator is

$$\bar{P}_0(x, t \mid x_0, t_0) = \{\exp[(t-t_0)\,\bar{L}^{(0)}]\}\;\delta(x-x_0) \tag{2.16}$$

which can be expressed in terms of the eigenfunctions of the unperturbed operator $\bar{L}^{(0)}$ as

$$\bar{P}_0(x, t \mid x_0, t_0) = \sum_n e^{-\lambda_n(t-t_0)}\,\bar{\psi}_n(x)\,\bar{\psi}_n(x_0) \tag{2.17}$$

When $(t-t_0)$ is very large, the conditional probability $\bar{P}$ can be obtained by plugging Eq. (2.17) and Eq. (2.11) into Eq. (2.15) and letting $t_0 \to -\alpha$. Noting that $\lambda_n > 0$, $\forall n > 0$, we express $\bar{P}(x, t \mid x_0, -\alpha)$ in terms of the eigenfunctions of the unperturbed operator $\bar{L}^{(0)}$ as

$$\begin{aligned}\bar{P}(x, t \mid x_0, -\alpha) = {} & \bar{\psi}_0(x)\,\bar{\psi}_0(x_0) - A\sum_n \bar{\psi}_n(x)\,\bar{\psi}_n(x_0)\langle n|\,\partial/\partial x - V_0'(x)/2D\,|0\rangle \\ & \times(\lambda_n^2+\Omega^2)^{-1/2}\cos(\Omega t+\theta+\alpha_n)\end{aligned} \tag{2.18}$$

where the phases $\alpha_n$ are defined through

$$\begin{aligned}\lambda_n &= (\lambda_n^2+\Omega^2)^{1/2}\cos\alpha_n \\ -\Omega &= (\lambda_n^2+\Omega^2)^{1/2}\sin\alpha_n\end{aligned} \tag{2.19}$$

The probability distribution $P(x, t)$ and the conditional probability distribution $P(x, t \mid x_0, t_0)$ are related by

$$P(x, t) = \int dx_0\, P(x, t \mid x_0, t_0)\, P(x_0, t_0)$$

and a similar relation exists for $\bar{P}$. Since $P(x, t)$ and $\bar{P}(x, t)$ are related by Eq. (2.12), the corresponding conditional probabilities are related by

$$P(x, t \mid x_0, t_0) = \bar{\psi}_0(x)\,\bar{P}(x, t \mid x_0, t_0)\,\bar{\psi}_0^{-1}(x_0) \tag{2.20}$$

With the help of Eqs. (2.20) and (2.18) one immediately gets the conditional probability for the original problem as

$$\begin{aligned}P(x, t \mid x_0, -\alpha) = {} & \bar{\psi}_0^2(x) - A\bar{\psi}_0(x)\sum_n \bar{\psi}_n(x)\langle n|\,\partial/\partial x - V_0'(x)/2D)\,|0\rangle \\ & \times(\lambda_n^2+\Omega^2)^{-1/2}\cos(\Omega t+\theta+\alpha_n)\end{aligned} \tag{2.21}$$

From the explicit expression of $\langle x|0\rangle$ it is immediately clear that

$$\langle n|\, \partial/\partial x\, |0\rangle = -\langle n|\, V_0'(x)/2D\, |0\rangle \tag{2.22}$$

When this result is substituted in Eq. (2.21) we obtain $P(x, t\,|\,x_0, -\alpha)$ as

$$P(x, t\,|\,x_0, -\alpha) = \bar{\psi}_0^2(x) + (A/D)\, \bar{\psi}_0(x) \sum_n \bar{\psi}_n(x) \langle n|\, V_0'\, |0\rangle \times (\lambda_n^2 + \Omega^2)^{-1/2} \cos(\Omega t + \theta + \alpha_n) \tag{2.23}$$

It is interesting to note that this expression is independent of the initial value $x_0$. We call it $P_\alpha(x, t)$. Thus the system evolving through Eq. (2.1) forgets its initial history after a considerable amount of time. All its moments are independent of its initial value after sufficient time has elapsed.

We now wish to calculate the spectral density or the power spectrum of the variable $X(t)$, since it will give directly the partition of the power at the output into signal and noise. By the Wiener–Khinchine theorem the power spectrum is the Fourier transform of the autocorrelation function and is given by

$$S(\omega, t) = \int \langle X(t)\, X(t+\tau)\rangle\, e^{i\omega\tau}\, d\tau \tag{2.24}$$

where $\langle X(t)\, X(t+\tau)\rangle$ is defined as the autocorrelation function

$$\langle X(t)\, X(t+\tau)\rangle = \iint dx\, dx_0\, x x_0 P(x, t+\tau; x_0, t) \tag{2.25}$$

where $P(x, t+\tau; x_0, t)$ is the joint probability of having the values of the stochastic variable $X$ at time $t$ as $x_0$ and at time $t+\tau$ as $x$. We wish to calculate this joint probability when $t$ is very large and sufficient time has elapsed after the system starts from some arbitrary value which would not appear in the distribution, as expected from the result (2.23). Thus $P(x, t+\tau;\, x_0, t)$ is given by

$$P(x, t+\tau; x_0, t) = P(x, t+\tau\,|\,x_0, t)\, P_\alpha(x_0, t) \tag{2.26}$$

The expression shows that we require $P(x, t+\tau\,|\,x_0, t)$ for arbitrary $\tau$. The procedure for evaluating this is exactly same as for $P_\alpha(x, t)$ except that we are not allowed to take the time difference arbitrarily large, which makes the algebra sufficiently simple. Hence $P(x, t+\tau\,|\,x_0, t)$ will necessarily contain many other terms. Here we make the following assumptions, which make the algebra less tedious. First, the potential $V_0(x)$ diverges very fast at least as $|x|^\alpha$ with $\alpha > 1$ as $|x| \to \infty$. This makes the separation between consecutive eigenvalues large and $\lambda_1 \ll \lambda_i$, $\forall i > 1$. Second, $V_0(x)$ is nonlinear and it possesses at least one maximum. For a bistable potential, for example, it has one maximum and two minima. We further assume that $V_0(x)$ is symmetric about $x = 0$, which simplifies some of the matrix elements in the derivation. Next, we know from Kramer's theory of the escape rate that the first nontrivial eigenvalue $\lambda_1$, which is nothing but the rate of escape from the potential well, is approximately proportional to the exponential decay of the ratio of barrier height to the diffusion constant. Later we will see that this feature provides a competitive characteristic between the noise kicks and periodic forcing on the system, leading to an optimum value of the barrier height or strength of the noise at which cooperation between them takes place.

Finally, we note from Eq. (2.24) that the power spectrum depends on time $t$. In experiment, however, $t$ is the time at which one starts to take data for calculating the spectrum. The time $t$ is arbitrary, hence one takes various values of $t$, computes the power spectrum, and then averages over all the spectra. Thus, if $t$ is taken with a uniform probability over the signal period $T = 2\pi/\Omega$, one obtains the average power spectrum $S(\omega)$ as

$$S(\omega) = (1/T) \int_0^T S(\omega, t)\, dt \tag{2.27}$$

This procedure simplifies the algebra considerably. We obtain the autocorrelation function averaged over signal period as

$$\overline{\langle X(t)\, X(t+\tau) \rangle} = [\Gamma_s \cos \Omega\tau + (1 - \Gamma_s)\, e^{-\lambda_1 \tau}] \langle 0 |\, x \,| 1 \rangle^2 \tag{2.28}$$

where the quantity $\Gamma_s$ is defined as

$$\Gamma_s = (A^2/2D^2) \langle 1 |\, V_0' \,| 0 \rangle^2 / (\lambda_1^2 + \Omega^2) \tag{2.29}$$

In the expression (2.28) we have omitted the terms $O(e^{-\lambda_2 \tau})$, $O(\lambda_3/[\lambda_3^2 + \Omega^2])$, and $O((\lambda_2 - \lambda_1)/[(\lambda_2 - \lambda_1)^2 + \Omega^2])$.

In the expression (2.28) we see that the autocorrelation function of the stochastic variable $X$, after sufficient time has elapsed, separates into two components. The first term, which is associated with $\cos \Omega\tau$, is oscillating with same frequency as the signal. Hence at the output we recognize this term as associated with the signal and the other term of course with the noise. As the noise term has an exponential dependence, it has a simple analogy with Ornstein–Uhlenbeck noise with correlation time $1/\lambda_1$. The term associated with the signal is proportional to $A^2$, hence it is small. That is, the power that is fed into the coherent signal is small. As $V_0(x)$ is symmetric, $\bar{L}^{(0)}(x)$ in Eq. (2.9) is also symmetric, hence the eigenfunctions have definite parities. $\bar{\psi}_0(x)$ is of course even and $\bar{\psi}_1(x)$ is odd. For a harmonic potential or for a linear force the matrix element $\langle 1|\, V_0'\, |0\rangle$ is zero. In order to have nontrivial $\Gamma_s$, a nonlinear force is required.

The power spectrum of the output is obtained by simply taking the Fourier transform of Eq. (2.28) and is given by

$$S(\omega) = \{\pi\Gamma_s[\delta(\omega+\Omega)+\delta(\omega-\Omega)] + (1-\Gamma_s)\, 2\lambda_1/(\lambda_1^2+\omega^2)\}\langle 0|\, x\, |1\rangle^2 \tag{2.30}$$

The signal-to-noise ratio at $\omega = \Omega$ (signal frequency) is then given by

$$I_o = \{\pi\Gamma_s/2(1-\Gamma_s)\}(\lambda_1^2+\Omega^2)/\lambda_1 = (\pi A^2/4D)\langle 1|\, V_0'\, |0\rangle^2/\lambda_1 D \tag{2.31}$$

where we have used the definition (2.29) of $\Gamma_s$ and smallness of $\Gamma_s$ with respect to unity. $I_o$ refers to the signal-to-noise ratio at the output.

The power spectrum of the noise and the signal at the input can be obtained similarly. They are $2D$ and $(\pi A^2/2)[\delta(\omega+\Omega)+\delta(\omega-\Omega)]$, respectively. Hence the signal-to-noise ratio at $\omega = \Omega$ is

$$I_i = \pi A^2/4D \tag{2.32}$$

Therefore, the signal-to-noise ratio at the output $I_o$ in Eq. (2.31) can be written in terms of the signal-to-noise ratio at the input as

$$I_o = I_i\langle 1|\, V_0'\, |0\rangle^2/\lambda_1 D \tag{2.33}$$

In order to have an idea about the dependence of $I_o$ on the potential barrier and diffusion constant one might attempt to calculate the expression (2.33). As $V_0$ is an arbitrary nonlinear function, the exact evaluation of $\lambda_1$ and $\langle x|1\rangle$ is not possible; however, one can evaluate them approximately.

As an illustration, we take the potential

$$V_0(x) = U_0[-2(x/c)^2 + (x/c)^4] \tag{2.34}$$

This bistable potential has one maximum at $x = 0$ and two minima situated symmetrically around $x = 0$ at $x = \pm c$. The barrier height of the above potential is $U_0$. With the approximate form of $\lambda_1$ and $\langle x|1\rangle$ for the potential (2.34) the signal-to-noise ratio at the output has been calculated in the recent literature.[7-9] The expression reads

$$I_o = I_i(4\sqrt{2}/\pi)(U_0/D)\, e^{-(U_0/D)} \tag{2.35}$$

The dependence of the output SNR on the noise strength $D$ can be seen by inserting the value of $I_i$ from Eq. (2.32) in Eq. (2.35),

$$I_o = \sqrt{2}A^2 U_0 D^{-2} e^{-(U_0/D)} \tag{2.36}$$

For small noise strength $D$ compared to the barrier height $U_0$, $I_o$ will be very small due to an exponentially small factor (small Kramer rate). On the other hand, for large values of $D$, it is again small due to the factor $D^{-2}$ (periodic forcing is dominated by noise kicks). Thus in between one obtains a maximum of the output SNR. This feature is counterintuitive in the sense that increasing noise strength from a very low value may cause the enhancement of the signal-to-noise ratio at the output. Looking at the dependence of $I_o$ on noise strength $D$, we find that the maximum value of $I_o$ occurs when $D = U_0/2$. Thus $(I_o)_{\max}$ becomes

$$(I_o)_{\max} = (4\sqrt{2}/e^2)(A^2/U_0) \tag{2.37a}$$

or, when expressed in terms of signal amplitude and noise strength,

$$(I_o)_{\max} = (2\sqrt{2}/e^2)(A^2/D) \tag{2.37b}$$

It is clear from Eq. (2.37b) that in order to have the maximum value of SNR at the output greater than 1 dB, one needs $A^2/D > 3.24$.

Our concern, however, is with the ratio of the output SNR to the input SNR, which can be seen from Eq. (2.35). We call this ratio $R$:

$$R = I_o/I_i = (4\sqrt{2}/\pi)\, x e^{-x} \tag{2.38}$$

where $x = U_0/D$. The ratio $R$ is independent of signal amplitude. For small and large $x$, $R$ will be small and one can see that the maximum of $R$ occurs when $x = 1$ or $D = U_0$:

$$(R)_{\max} = (4\sqrt{2}/\pi e) \simeq 0.66 \tag{2.39}$$

This result shows that for the nonlinear bistable system driven by a periodic signal in a white-noise environment, the ratio of the output SNR to the input SNR is always less than unity.

## 3. ENHANCEMENT OF THE RATIO OF OUTPUT SNR TO INPUT SNR

We next consider a nonlinear system perturbed stochastically by a harmonic noise. Signal is applied at the source of the noise generator. This system recently has been studied experimentally.[15] In this section we analyze the system theoretically. The system is described by the following set of Langevin equations:

$$\dot{x} = -V_0'(x) + ky \tag{3.1}$$

$$\dot{y} = Y \tag{3.2}$$

$$\dot{Y} = -\gamma Y + \omega_0^2 y + A\cos(\Omega t + \theta) + \Gamma(t) \tag{3.3}$$

where $k$ is a parameter of the linear amplifier; $\gamma$, $\omega_0$ are the parameters of the harmonic noise generator; $A$ and $\Omega$ are the characteristics of the signal, as mentioned before; and $\Gamma(t)$ is white with the properties (2.2a) and (2.2b).

The Fokker–Planck equation corresponding to Eq. (3.1) is

$$\frac{\partial P(x, y, Y, t)}{\partial t} = \Gamma P(x, y, Y, t) \tag{3.4}$$

where $P(x, y, Y, t)$ is the probability distribution for the full description of the system and the operator $\Gamma$ is broken up into parts:

$$\Gamma = \Gamma_0 + \Gamma_1 \tag{3.5}$$

with $\Gamma_0$ given by

$$\Gamma_0 = -Y\frac{\partial}{\partial y} + \frac{\partial}{\partial Y}\left(\gamma Y + \omega_0^2 y + D\frac{\partial}{\partial Y}\right) \tag{3.6}$$

$\Gamma_1$ is again broken up into an unperturbed part $\Gamma_1^{(0)}$ and a perturbed part $\Gamma_1^{(1)}$. All these partitions are done for the convenience of later developments. We have

$$\Gamma_1 = \Gamma_1^{(0)} + \Gamma_1^{(1)} \tag{3.7}$$

where $\Gamma_1^{(0)}$ and $\Gamma_1^{(1)}$ are

$$\Gamma_1^{(0)} = -\frac{\partial}{\partial x}[-V_0'(x) + ky] \tag{3.8}$$

$$\Gamma_1^{(1)} = -A\cos(\Omega t + \theta)\frac{\partial}{\partial Y} \tag{3.9}$$

We note that in this partition $\Gamma_0$ contains coefficients which are at best linear and this operator has been well studied.[16] Perturbation by the signal is only through the operator $\Gamma_1^{(1)}$. The remaining unperturbed part consists, however, of two operators. We denote the unperturbed operator by $\Gamma^{(0)}$:

$$\Gamma^{(0)} = \Gamma_0 + \Gamma_1^{(0)} \tag{3.10}$$

Equation (3.4) can be transformed into a more convenient form by defining a function $\bar{f}(x, y, Y, t)$ related to $P(x, y, Y, t)$ by

$$\bar{f}(x, y, Y, t) = e^{\Phi(y, Y)/2} P(x, y, Y, t) \tag{3.11}$$

where $\Phi(y, Y)$ is given by

$$\Phi(y, Y) = (\gamma/2D)(Y^2 + \omega_0^2 y^2) \tag{3.12}$$

The function $\bar{f}(x, y, Y, t)$ then satisfies the following evolution equation:

$$\frac{\partial \bar{f}(x, y, Y, t)}{\partial t} = \bar{\Gamma}\bar{f}(x, y, Y, t) \tag{3.13}$$

where $\bar{\Gamma}$ is related to $\Gamma$ by

$$\bar{\Gamma} = e^{\Phi/2}\Gamma e^{-\Phi/2} \tag{3.14}$$

The operator $\bar{\Gamma}$ consists of $\bar{\Gamma}^{(0)}$ and $\bar{\Gamma}_1^{(1)}$, which are related to $\Gamma^{(0)}$ and $\Gamma_1^{(1)}$, respectively, by relations similar to (3.14). They explicitly read as

$$\bar{\Gamma}^{(0)} = \bar{\Gamma}_0 + \bar{\Gamma}_1^{(0)} \tag{3.15}$$

$$\bar{\Gamma}_0 = -\omega_0(ab^\dagger - a^\dagger b) - \gamma b^\dagger b \tag{3.16}$$

$$\bar{\Gamma}_1^{(0)} = -\frac{\partial}{\partial x}[-V_0'(x) + ky] \tag{3.17}$$

$$\bar{\Gamma}_1^{(1)} = -(A/\sqrt{D/\gamma})\cos(\Omega t + \theta)\, b^\dagger \tag{3.18}$$

where the operators $a$, $b$ are defined by

$$a = (\sqrt{D/\gamma}/\omega_0)\,\partial/\partial y + (\omega_0/2\sqrt{D/\gamma})\, y \tag{3.19}$$

$$b = \sqrt{D/\gamma}\,\partial/\partial Y + (1/2\sqrt{D/\gamma})\, Y \tag{3.20}$$

The corresponding adjoint oprators $a^\dagger, b^\dagger$ are defined similarly. The operators $a$, $a^\dagger$, $b$, $b^\dagger$ satisfy the commutation properties[16]

$$[a, a^\dagger] = [b, b^\dagger] = 1 \tag{3.21}$$

$$[a, b] = [a^\dagger, b] = [a, b^\dagger] = [a^\dagger, b] = 0 \tag{3.22}$$

The operator $\bar{\Gamma}_0$ is sufficiently simple, although it is not Hermitian. It possesses a complete set of normalized eigenfunctions $\bar{\psi}_{n_1,n_2}(y, Y)$ with the corresponding eigenvalues $-\lambda_{n_1,n_2}$,[16]

$$\bar{\Gamma}_0 \bar{\psi}_{n_1,n_2}(y, Y) = -\lambda_{n_1,n_2} \bar{\psi}_{n_1,n_2}(y, Y) \tag{3.23}$$

$$\bar{\Gamma}_0^\dagger \bar{\psi}_{n_1,n_2}^\dagger(y, Y) = -\lambda_{n_1,n_2} \bar{\psi}_{n_1,n_2}^\dagger(y, Y) \tag{3.24}$$

$$\int dy\, dY\, \bar{\psi}_{n_1,n_2}^\dagger(y, Y)\, \bar{\psi}_{n_1',n_2'}(y, Y) = \delta_{n_1,n_1'} \delta_{n_2,n_2'} \tag{3.25}$$

We note that as $\bar{\Gamma}_0 \neq \bar{\Gamma}_0^\dagger$, one has to invoke the eigenfunctions of the adjoint operator $\bar{\Gamma}_0^\dagger$. The eigenfunction of $\bar{\Gamma}_0^\dagger$ corresponding to the eigenvalue $-\lambda_{n_1,n_2}$ is denoted $\psi_{n_1,n_2}^\dagger(y, Y)$ and the eigenvalues are

$$\lambda_{n_1,n_2} = n_1 \lambda_1 + n_2 \lambda_2; \qquad n_1, n_2 \in \mathbb{Z} \tag{3.26}$$

with $\lambda_1, \lambda_2$ being the eigenvalues of the matrix

$$\begin{pmatrix} 0 & -1 \\ \omega_0^2 & \gamma \end{pmatrix} \tag{3.27}$$

The eigenvalues $\lambda_1, \lambda_2$ satisfy the properties

$$\lambda_1 + \lambda_2 = \gamma \tag{3.28a}$$

$$\lambda_1 \lambda_2 = \omega_0^2 \tag{3.28b}$$

$$\mathrm{Re}(\lambda_1, \lambda_2) > 0 \tag{3.28c}$$

Of particular interest is the eigenvalue $\lambda_{0,0}$, which is equal to zero, and the eigenfunctions $\bar{\psi}_{0,0}(y, Y)$ and $\bar{\psi}_{0,0}^\dagger(y, Y)$ corresponding to $\bar{\Gamma}_0$ and $\bar{\Gamma}_0^\dagger$ are the same.

Equation (3.13) describes the time development of the probability distribution, which is a function of $x, y, Y$, and $t$. However, our concern is with the time development of the reduced distribution, which is a function of stochastic variables $x$ and time $t$ only. Although the full description is Markovian in nature, the integrated or reduced description naturally depends on the history, making the description non-Markovian in nature. Our aim is to obtain a Markov description of the reduced distribution under suitable approximation. The coarse-grained distribution at any time $t$ would be connected to its previous values through the nontrivial eigenvalues $\lambda_{n_1,n_2}$ $(n_1, n_2 \neq 0)$; the contributions of these terms would be expected to decay exponentially with the decay constant $\gamma$. As these terms constitute the history which makes the description non-Markovian, this suggests we define a projection operator $P$ which projects out the full description into the $\bar{\psi}_{0,0}(y, Y)$ and look for the time development of the projected distribution function. The operator $P$ is defined to be

$$P\,|\bar{f}\rangle = |0, 0\rangle\langle 0, 0\,|\,\bar{f}\rangle_{x,t} \tag{3.29}$$

The integer numbers in the ket correspond to $n_1, n_2$ of $\lambda_{n_1,n_2}$. The integers in the bra correspond to the eigenfunctions of the adjoint operator $\bar{\Gamma}_0^\dagger$ and the projection is taken at each value of $x$ and $t$.

Our next development parallels the projection operator method due to Nakajima, Zwanzig, and Mori.[17]

We start by writing the time development equations for the projected distribution and its complementary part:

$$\frac{\partial(P\bar{f})}{\partial t} = P\bar{\Gamma}(P\bar{f}) + P\bar{\Gamma}(P'\bar{f}) \tag{3.30}$$

$$\frac{\partial(P'\bar{f})}{\partial t} = P'\bar{\Gamma}(P'\bar{f}) + P'\bar{\Gamma}(P\bar{f}) \tag{3.31}$$

where the operator $P'$ projects the distribution onto the space complementary to the zeroth eigenstate such that

$$P + P' = 1 \tag{3.32}$$

Our intention is to obtain the time development equation for $P\bar{f}$. Hence we solve Eq. (3.31) for obtaining the solution $P'\bar{f}(t)$, which will be substituted back into Eq. (3.30) to obtain the required equation.

Given the values of the distribution $\bar{f}$ at time $t_0$, we can write the solution $P'\bar{f}(t)$ formally as

$$\begin{aligned} P'\bar{f}(t) &= e^{(t-t_0)P'\bar{\Gamma}^{(0)}}P'\bar{f}(t_0) + \int_{t_0}^{t} d\tau\, e^{(t-\tau)P'\bar{\Gamma}^{(0)}}P'\bar{\Gamma}P\bar{f}(\tau) \\ &\quad + \int_{t_0}^{t} d\tau\, e^{(t-\tau)P'\bar{\Gamma}^{(0)}}P'\bar{\Gamma}_1^{(1)}[P'\bar{f}(\tau)] \end{aligned} \tag{3.33}$$

where $\bar{\Gamma}$ on the r.h.s. of Eq. (3.31) is broken up into an unperturbed part $\bar{\Gamma}^{(0)}$ [defined in Eq. (3.15)] and an explicitly time-dependent part $\bar{\Gamma}_1^{(1)}(t)$ [defined in Eq. (3.18)].

The following properties of $P$ simplify the derivation considerably:

$$\bar{\Gamma}_0 P = 0 \tag{3.34}$$

$$P\bar{\Gamma}_0 = 0 \tag{3.35}$$

The first equality (3.34) is obtained since $\lambda_{0,0}=0$. The second equality (3.35) results because $|0, 0\rangle$ is the eigenstate for both the operators $a$ and $b$ with zero as the eigenvalue.

The first term on the r.h.s. of Eq. (3.33) describes the evolution of the initial distribution $\bar{f}(t_0)$. We will consider this term later. Let us concentrate on the second term of the r.h.s. of Eq. (3.33). We note that $\bar{\Gamma}^{(0)}$ appearing in the exponent consists of two parts,

$$\bar{\Gamma}^{(0)} = \bar{\Gamma}_0 + \bar{\Gamma}_1^{(0)} \tag{3.15}$$

As $\bar{\Gamma}_0$ involves $\gamma$ and $\bar{\Gamma}_1^{(0)}$ does not, we have for large $\gamma$,

$$e^{(t-\tau)P'\bar{\Gamma}^{(0)}} \simeq e^{(t-\tau)P'\bar{\Gamma}_0} \tag{3.36}$$

and with the help of Eqs. (3.32) and (3.35) one writes

$$e^{(t-\tau)P'\bar{\Gamma}_0} = e^{(t-\tau)\bar{\Gamma}_0} \tag{3.37}$$

Next, with the help of Eq. (3.32) one sees

$$P'\bar{\Gamma}P\bar{f} = \bar{\Gamma}P\bar{f} - P\bar{\Gamma}P\bar{f} \tag{3.38}$$

where the term $P\bar{\Gamma}P\bar{f}$, which arises also in Eq. (3.30), simplifies on using Eq. (3.35) to

$$\begin{aligned} P\bar{\Gamma}P\bar{f}(t) = \bar{\psi}_{0,0}(y, Y)\bigg\{ -\frac{\partial}{\partial x}[-V_0'(x)\,g(x,t)] - k\langle 0,0|\,y\,|0,0\rangle \frac{\partial g(x,t)}{\partial x} \\ + (A/\sqrt{D/\gamma})\cos(\Omega t+\theta)\,\langle 0,0|\,b^{+}\,|0,0\rangle\, g(x,t)\bigg\} \end{aligned} \tag{3.39}$$

The quantity $g(x, t)$ in Eq. (3.39) refers to the reduced distribution and is defined as

$$g(x,t) = \langle 0,0\,|\,\bar{f}\rangle_{x,t} \tag{3.40}$$

With the help of Eqs. (3.34) and (3.37), the second term of r.h.s. of Eq. (3.33) simplifies to

$$\begin{aligned} \int_{t_0}^{t} d\tau\, e^{(t-\tau)\bar{\Gamma}_0}\bigg\{ -k\frac{\partial g(x,\tau)}{\partial x}\,(y\bar{\psi}_{0,0}(y, Y)) \\ + (A/\sqrt{D/\gamma})\cos(\Omega\tau+\theta)\, g(x,\tau)(b^{+}\bar{\psi}_{0,0}(y, Y))\bigg\} \end{aligned} \tag{3.41}$$

As $\bar{\Gamma}_0$ appears on the left in the exponent of Eq. (3.41), one should express $(y\bar{\psi}_{0,0}(y, Y))$ and $(b^\dagger\bar{\psi}_{0,0}(y, Y))$ in terms of the eigenfunctions of the operator $\bar{\Gamma}_0$. The results are

$$y\bar{\psi}_{0,0}(y, Y) = (\sqrt{D/\gamma}/\omega_0)\,\delta^{-1/2}(\sqrt{\lambda_2}\bar{\psi}_{1,0} + \sqrt{\lambda_1}\bar{\psi}_{0,1}) \tag{3.42}$$

$$b^\dagger\bar{\psi}_{0,0}(y, Y) = \delta^{-1/2}(\sqrt{\lambda_1}\bar{\psi}_{1,0} + \sqrt{\lambda_2}\bar{\psi}_{0,1}) \tag{3.43}$$

where $\delta = \lambda_1 - \lambda_2$.

We next look at the r.h.s. of Eq. (3.30). The first term is already evaluated in Eq. (3.39). As $y\bar{\psi}_{0,0}(y, Y)$ and $b^\dagger\bar{\psi}_{0,0}(y, Y)$ contain functions like $\bar{\psi}_{1,0}$ and $\bar{\psi}_{0,1}$, by employing the orthogonality relation (3.25), we find that Eq. (3.39) further simplifies to

$$P\bar{\Gamma}P\bar{f}(t) = \bar{\psi}_{0,0}(y, Y)\left\{-\frac{\partial}{\partial x}[-V_0'(x)\,g(x, t)]\right\} \tag{3.44}$$

We see from Eq. (3.33) that $P'\bar{f}$ consist of three terms. The second term of Eq. (3.33) is simplified to Eq. (3.41), on which $P\bar{\Gamma}$ is to be operated while consisdering Eq. (3.30). One may notice $P\bar{\Gamma} = P\bar{\Gamma}_1$ because of Eq. (3.35) and $P\bar{\Gamma}_1 = P(\bar{\Gamma}_1^{(0)} + \bar{\Gamma}_1^{(1)})$ from the definition (3.7). The operator $P\bar{\Gamma}_1^{(1)}$ operating on any function will be identically zero because of the fact that $b\bar{\psi}_{0,0}^\dagger = 0$. Next, $\bar{\Gamma}_1^{(0)}$ consists of two parts. As (3.41) contains the functions $\bar{\psi}_{1,0}$, $\bar{\psi}_{0,1}$, the term associated with $-(\partial/\partial x)[-V_0'(x)]$ will be dropped due to orthogonalities of these functions with $\bar{\psi}_{0,0}^\dagger$ in the operator $P$. Hence we will be left with the term associated with $-ky'\,\partial/\partial x$, i.e.,

$$P\bar{\Gamma}_1(3.41) = -k\frac{\partial}{\partial x}\langle y, Y|0, 0\rangle\langle 0, 0|\, y' * (3.41)\rangle \tag{3.45}$$

Since $(y\bar{\psi}_{0,0})$ contains $\bar{\psi}_{1,0}$ and $\bar{\psi}_{0,1}$ and (3.41) also contains the same functions, the r.h.s. of Eq. (3.45) will be nontrivial. This term can be calculated in a straightforward manner as

$$\begin{aligned}\bar{\psi}_{0,0}(y, Y)\int_{t_0}^{t} d\tau &\left\{\frac{k^2 D}{\gamma\omega_0^2\delta}\frac{\partial}{\partial x}\left(\frac{\partial g(x, \tau)}{\partial x}\right)(-\lambda_2 e^{-(t-\tau)\lambda_1} + \lambda_1 e^{-(t-\tau)\lambda_2})\right.\\ &\left.-\frac{Ak}{\delta}\cos(\Omega\tau + \theta)\left(\frac{\partial g(x, \tau)}{\partial x}\right)(-e^{-(t-\tau)\lambda_1} + e^{-(t-\tau)\lambda_2})\right\}\end{aligned} \tag{3.46}$$

The expression (3.46) shows that the time development of the reduced distribution $g(x, t)$ depends on the earlier history. However, we note that the maximum contribution of (3.46) comes from the time $\tau \simeq t$. The term $e^{-(t-\tau)\lambda_1}$ or $e^{-(t-\tau)\lambda_2}$ decays very fast as $t-\tau > 1/\gamma$. As has been stated, we would like to obtain a Markovised description of the reduced distribution $g(x, t)$. Markovization implies that we replace $g(x, \tau)$ in the integrand by $g(x, t)$. This means that the fractional change of $g(x, t)$ over a time $1/\gamma$ will be less than unity,

$$|(1/g)\, \partial g/\partial t|/\gamma \ll 1 \tag{3.47}$$

Finally, as the integrand decays very fast due to the exponential factor, we can replace the lower limit of the integral $t_0$ by $-\alpha$. With these substitutions, (3.46) is simplified to

$$\bar{\psi}_{0,0}(y, Y)\left[D\left(\frac{k}{\omega_0^2}\right)^2 \frac{\partial^2 g(x, t)}{\partial x^2} - kA_0 \cos(\Omega t + \theta + \beta)\, \frac{\partial g(x, t)}{\partial x}\right] \tag{3.48}$$

where

$$A_0 = A[(\lambda_1^2 + \Omega^2)(\lambda_2^2 + \Omega^2)]^{-1/2} = A[\gamma^2\Omega^2 + (\omega_0^2 - \Omega^2)^2]^{-1/2} \tag{3.49}$$

and the phase $\beta$ is defined through

$$\tan\beta = -\gamma\Omega/(\omega_0^2 - \Omega^2) \tag{3.50}$$

We will now consider the fist term in Eq. (3.33) describing the evolution of the initial distribution $\bar{f}(t_0)$. The term is

$$e^{(t-t_0)\bar{\Gamma}_0} P' \bar{f}(t_0) \tag{3.51}$$

where we have used the results of Eqs. (3.36)–(3.37). In order to evaluate the term, we expand $P'\bar{f}(t_0)$ in terms of the eigenfunctions of $\bar{\Gamma}_0$. Since $\{|n_1, n_2\rangle\}$ is complete, one writes (3.51) explicitly as

$$\sum_{n_1, n_2}{}' \{\exp[-\lambda_{n_1, n_2}(t - t_0)]\}\, |n_1, n_2\rangle\langle n_1, n_2|\bar{f}\rangle \tag{3.52}$$

where the prime on the summation indicates the exclusion of $n_1 = n_2 = 0$. Because $\mathrm{Re}\{\lambda_{n_1, n_2}\} > 0$, $\forall n_1, n_2 > 0$, for $(t - t_0) > 1/\gamma$ the contribution decays very fast, so that the effect of the initial distribution can be ignored in this time regime.

We have not still considered the third term in Eq. (3.33). If we ignore the third term, $P'\bar{f}(t)$ will be equal to (3.41) in the specified time regime, where the contribution from the initial distribution can be ignored. When written explicitly, Eq. (3.33) takes the form

$$P'\bar{f}(t) = [P'\bar{f}(t)]^{(0)} + \int_{t_0}^{t} d\tau\, e^{(t-\tau)\bar{\Gamma}_0} P'\bar{\Gamma}_1^{(1)}[P'\bar{f}(\tau)] \tag{3.53}$$

where we have used the notation $[P'\bar{f}(t)]^{(0)}$ for (3.41) and used the results of (3.36)–(3.37).

Equation (3.53) can be thought of as an integral equation in $[P'\bar{f}(t)]$, where the zeroth-order approximation is written as $[P'\bar{f}(t)]^{(0)}$. Introducing the operator $K(t, \tau)$, we write Eq. (3.53) more explicitly as

$$[P'\bar{f}(t)] = [P'\bar{f}(t)]^{(0)} + \int_{t_0}^{t} d\tau\, K(t, \tau)[P'\bar{f}(\tau)] \tag{3.54}$$

where $K(t, \tau)$ is defined as

$$K(t, \tau) = e^{(t-\tau)\bar{\Gamma}_0} P'\bar{\Gamma}_1^{(1)} \tag{3.55}$$

From our earlier discussion we saw that $P\bar{\Gamma}_1^{(1)} \equiv 0$. Hence $K(t, \tau)$ is further simplified to

$$K(t, \tau) = e^{(t-\tau)\bar{\Gamma}_0} \bar{\Gamma}_1^{(1)} \tag{3.56}$$

The first-order approximation of $[P'\bar{f}(t)]$ is obtained by substituting $[P'\bar{f}(\tau)]$ in the integral of (3.54) by $[P'\bar{f}(t)]^{(0)}$:

$$[P'\bar{f}(t)]^{(1)} = [P'\bar{f}(t)]^{(0)} + \int_{t_0}^{t} d\tau\, K(t, \tau)[P'\bar{f}(\tau)]^{(0)} \tag{3.57}$$

Consider the integral in Eq. (3.57). We have seen from Eqs. (3.42)–(3.43) that $[P'\bar{f}(t)]^{(0)}$ contains the functions $\bar{\psi}_{1,0}$, $\bar{\psi}_{0,1}$. When they are acted on by $\bar{\Gamma}_1^{(1)}(\tau)$, which contains the $b^\dagger$ operator, they produce the functions $\bar{\psi}_{2,0}$, $\bar{\psi}_{1,1}$, $\bar{\psi}_{0,2}$. As these functions are eigenfunctions of $\bar{\Gamma}_0$, the integral in Eq. (3.57) consists of these functions only. The expression $[P'\bar{f}(t)]^{(1)}$ thus obtained is to be substituted in Eq. (3.30) in order to get the time evolution for projected distribution. The r.h.s. of Eq. (3.30) where $[P'\bar{f}(t)]^{(1)}$ is to be substituted is

$$P\bar{\Gamma}[P'\bar{f}(t)]^{(1)} = P\bar{\Gamma}_1[P'\bar{f}(t)]^{(1)} = P\bar{\Gamma}_1^{(0)}[P'\bar{f}(t)]^{(1)} \tag{3.58}$$

The operator $P$ defines the inner product with $\bar{\psi}_{0,0}^\dagger$. So the terms associated with $-(\partial/\partial x)(-V_0'(x))$ in $\bar{\Gamma}_1^{(0)}$ drop out because of the orthogonality of $\bar{\psi}_{2,0}$, $\bar{\psi}_{1,1}$, and $\bar{\psi}_{0,2}$ with $\bar{\psi}_{0,0}$. As $(y\bar{\psi}_{0,0}^\dagger)$ arising from the term $-ky'\ \partial/\partial x$

in $\bar{\Gamma}_1^{(0)}$ contains the functions $\bar{\psi}_{1,0}^{\dagger}$ and $\bar{\psi}_{0,1}^{\dagger}$, again due to their orthogonality with $\bar{\psi}_{2,0}$, $\bar{\psi}_{1,1}$, and $\bar{\psi}_{0,2}$, the term will be identically zero, leading to the result

$$P\bar{\Gamma}[P'\bar{f}(t)]^{(1)} = P\bar{\Gamma}[P'\bar{f}(t)]^{(0)} \tag{3.59}$$

Before we go over to the higher iterates of $[P'\bar{f}(t)]$, we make the following observations:

(1) $[P'\bar{f}(t)]^{(0)}$ consists of $\bar{\psi}_{1,0}$, $\bar{\psi}_{0,1}$.

(2) $\int_{t_0}^{t} d\tau\, K(t,\tau)[P'\bar{f}(\tau)]^{(0)}$ consists of $\bar{\psi}_{2,0}$, $\bar{\psi}_{1,1}$, $\bar{\psi}_{0,2}$, i.e., actions of $K(t,\tau)$ produce all possible eigenstates of $\bar{\Gamma}_0$ one unit higher than the eigenfunctions contained in the functions on which it acts.

(3) The effective action of the oprator $P\bar{\Gamma}$ is to take inner products of $\bar{\psi}_{0,0}^{\dagger}$, $\bar{\psi}_{1,0}^{\dagger}$, and $\bar{\psi}_{0,1}^{\dagger}$ with the functions on which it acts.

It is clear that for the second iterate of $P'\bar{f}(t)$ one has

$$P\bar{\Gamma}[P'\bar{f}(t)]^{(2)} = P\bar{\Gamma}[P'\bar{f}(t)]^{(1)} + P\bar{\Gamma}\int_{t_0}^{t} d\tau\, K(t,\tau)[P'\bar{f}(\tau)]^{(1)} \tag{3.60}$$

which, with the result (3.59), boils down to

$$P\bar{\Gamma}[P'\bar{f}(t)]^{(2)} = P\bar{\Gamma}[P'\bar{f}(t)]^{(0)} + P\bar{\Gamma}\int_{t_0}^{t} d\tau\, K(t,\tau)[P'\bar{f}(\tau)]^{(1)} \tag{3.61}$$

Since $[P'\bar{f}(t)]^{(1)}$ consists of $\bar{\psi}_{1,0}$, $\bar{\psi}_{0,1}$, $\bar{\psi}_{2,0}$, $\bar{\psi}_{1,1}$, and $\bar{\psi}_{0,2}$, according to observation (2), $\int_{t_0}^{t} d\tau\, K(t,\tau)[P'\bar{f}(t)]^{(1)}$ contains the functions $\bar{\psi}_{2,0}$, $\bar{\psi}_{1,1}$, $\bar{\psi}_{0,2}$, $\bar{\psi}_{3,0}$, $\bar{\psi}_{2,1}$, $\bar{\psi}_{1,2}$, and $\bar{\psi}_{0,3}$ and in accordance with observation (3),

$$P\bar{\Gamma}\int_{t_0}^{t} d\tau\, K(t,\tau)[P'\bar{f}(\tau)]^{(1)} = 0 \tag{3.62}$$

Thus we get

$$P\bar{\Gamma}[P'\bar{f}(t)]^{(2)} = P\bar{\Gamma}[P'\bar{f}(t)]^{(0)} \tag{3.63}$$

In this way we can see that in Eq. (3.30) we need to consider no higher iterates of $[P'\bar{f}]$ than the zeroth order. Hence in the Markov approximation, the time evolution equation for projected distribution $g(x,t)$ turns out to be

$$\frac{\partial g(x,t)}{\partial t} = [L_{\mathrm{M}}^{(0)} + L_{\mathrm{M}}^{(1)}(t)]\, g(x,t) \tag{3.64}$$

where $L_{\mathrm{M}}^{(0)}$ and $L_{\mathrm{M}}^{(1)}(t)$ are obtained from Eqs. (3.44) and (3.48). They are

$$L_{\mathrm{M}}^{(0)} = -\frac{\partial}{\partial x}(-V_0'(x)) + D\left(\frac{k}{\omega_0^2}\right)^2 \frac{\partial^2}{\partial x^2} \tag{3.65}$$

$$L_{\mathrm{M}}^{(1)}(t) = -kA_0 \cos(\Omega t + \theta + \beta) \frac{\partial}{\partial x} \tag{3.66}$$

Once we obtain the Markovized description of the process (3.4), we can use the result (2.31) for the expression of the output SNR at $\omega = \Omega$. The only substitutions that we have to make in (2.31) are

$$A \to kA_0 \tag{3.67}$$

$$D \to D(k/\omega_0^2)^2 \tag{3.68}$$

The signal-to-noise ratio at the output at the signal frequency is

$$I_o = (\pi A^2/4D)\{\omega_0^4/[\gamma^2\Omega^2 + (\omega_0^2 - \Omega^2)^2]\}\langle 1|\ V_0'\ |0\rangle^2/\lambda_1 D \tag{3.69}$$

As the signal-to-noise ratio at the input is

$$I_i = \pi A^2/4D \tag{3.70}$$

the ratio of output SNR to input SNR is

$$R = I_o/I_i = \{\omega_0^4/[\gamma^2\Omega^2 + (\omega_0^2 - \Omega^2)^2]\}\langle 1|\ V_0'\ |0\rangle^2/\lambda_1 D \tag{3.71}$$

Comparing the expression (3.71) with (2.33), we note that the ratio obtained from (2.33) is modified by a factor

$$\omega_0^4/[\gamma^2\Omega^2 + (\omega_0^2 - \Omega^2)^2] \tag{3.72}$$

This factor assumes a particularly simple form when the characteristic of the harmonic noise $\omega_0$ matches with the signal frequency $\Omega$. In this special case the factor in (3.72) turns out to be $Q^2$, where $Q$ is called the quality factor and is defined as

$$Q = \Omega/\gamma \tag{3.73}$$

From our previous analysis we found that for a bistable potential the ratio $R$ takes a maximum value roughly equal to 0.66 [see Eq. (2.39)]. In the process (3.4), where the potential $V_0$ is assumed to be bistable and the noise characteristic $\omega_0$ matches with the signal frequency $\Omega$, the maximum value of the ratio $R$ changes to $0.66Q^2$, which could be made greater than unity with the improvement of the quality factor of the linear filter.

## 4. CONCLUDING REMARKS

We have shown that the ratio of output SNR to input SNR is less than unity when a bistable system is perturbed by a signal in a white-noise environment. We further show that when a nonlinear system is perturbed by harmonic noise and the signal is applied at the source of the harmonic noise generator, this ratio can be made larger than one. This feature makes the enhancement of the SNR at the output much better than with the white-noise case. Hence the quality of the signal can be improved considerably.

This analysis shows that output SNR or ratio of output SNR to input SNR is independent of the noise parameter $k$, which is the amplification factor in the linear amplifier.

If the signal is introduced on the r.h.s. of Eq. (3.1) and not at the source of the noise generator as in (3.3), the corresponding time evolution equation in the Markov approximation would be identical to Eq. (3.64) except that the operator $L_{\mathrm{M}}^{(1)}$ would be

$$L_{\mathrm{M}}^{(1)}(t) = -A\cos(\Omega t + \theta)\frac{\partial}{\partial x} \tag{4.1}$$

The ratio of output SNR to input SNR at the signal frequency would be in this case

$$R = (\omega_0^2/k)^2 \langle 1 |\, V_0' \,| 0 \rangle^2 / \lambda_1 D \tag{4.2}$$

Thus in this situation the modification factor would depend on the amplification factor $k$, although it would become independent of the parameter $\gamma$.

Finally, instead of harmonic noise, if one applies Ornstein–Uhlenbeck noise (OU), the Langevin equation reads

$$\dot{x} = -V_0'(x) + ky \tag{4.3}$$

$$\dot{y} = -\gamma Y + A\cos(\Omega t + \theta) + \Gamma(t) \tag{4.4}$$

where $\Gamma(t)$ is stationary white noise. It can be shown by a similar analysis that the time evolution of the projected distribution in the Markov approximation is

$$\frac{\partial g(x,t)}{\partial t} = [L_{\mathrm{M}}^{(0)} + L_{\mathrm{M}}^{(1)}(t)]\, g(x,t) \tag{4.5}$$

where $L_{\mathrm{M}}^{(0)}$ and $L_{\mathrm{M}}^{(1)}(t)$ are defined as

$$L_{\mathrm{M}}^{(0)} = -(\partial/\partial x)\,(-V_0'(x)) + D(k/\gamma)^2\,\partial^2/\partial x^2 \tag{4.6}$$

$$L_{\mathrm{M}}^{(1)}(t) = -k[A/(\gamma^2+\Omega^2)^{1/2}]\cos(\Omega t+\theta+\alpha)\,\partial/\partial x \tag{4.7}$$

with the phase $\alpha$ be determined through

$$\tan\alpha = -\Omega/\gamma \tag{4.8}$$

The ratio of output SNR to input SNR at the signal frequency would be in this case

$$R = [\gamma^2/(\gamma^2+\Omega^2)]\langle 1|\,V_0'\,|0\rangle^2/\lambda_1 D \tag{4.9}$$

For a bistable potential the maximum value of $\langle 1|\,V_0'\,|0\rangle^2/\lambda_1 D$ is approximately 0.66, and as the modification factor is less than unity for finite $\Omega$, the ratio which is of primary concern in this paper is less than unity when the nonlinear system is subject to a noisy environment having the characteristic of OU noise.

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